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THE CURVATURE GROUPS OF A PSEUDO-RIEMANNIAN MANIFOLD

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1. Introduction

A cochain complex associated with the Levi-Civita connection Γ of an n-dimensional (pseudo-) Riemannian manifold (M, γ) with metric γ is introduced. Its cohomology groups $H^p(M, \Gamma)$, $p = 1, \cdots, n$, called the curvature groups, are investigated, and it is shown that they are isomorphic with the cohomology groups $H^p(M, \mathscr{S})$ of M with coefficients in a subsheaf $\mathscr{S}$ of the sheaf of germs of infinitesimal homothetic transformations of M. This extends the principal result of I. Vaisman [2] concerning locally flat manifolds. The covariant form of the elements of $\mathscr{S}$ defined by duality in terms of the metric γ are closed. Curvature is introduced by means of the integrability conditions of the differential system defining the elements of $\mathscr{S}$. As a consequence, if the Ricci tensor is nondegenerate everywhere, then the curvature groups vanish. In particular, if γ is an Einstein metric and at least one of the curvature groups is not trivial, then it is Ricci flat. More generally, if the scalar curvature is a nonzero constant, but (M, γ) is not necessarily an Einstein space, then the curvature groups are isomorphic with the cohomology groups of M with coefficients in the sheaf of germs of its parallel vector fields. On the other hand, if $\mathscr{S}$ is not empty and there are no parallel vector fields (locally), then the groups $H^p(M, \Gamma)$ are isomorphic with the corresponding de Rham groups of M.

2. Tensorial p-forms

Let $P(M, G)$ be a principal fibre bundle over M with group G, Γ a connection in P, E a finite dimensional vector space, and ρ a linear representation of G in E.

A tensorial p-form, $p \geq 1$, of type $\rho(G)$ is a p-form φ on P with values in E satisfying the following conditions:

(i) $\varphi(X_1, \cdots, X_p) = 0$, whenever at least one of the $X_i \in T_u(P)$, $i = 1, \cdots, p$, is vertical;

(ii) $\varphi(R_{g*}X_1, \cdots, R_{g*}X_p) = \rho^{-1}(g)\varphi(X_1, \cdots, X_p)$, $\forall g \in G$ where R_{g*} de-

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notes the linear mapping induced on the tangent space $T_u(P)$ by the right translation R_g by which G operates on P.

For $p = 0$ we have a tensor of type $\rho(G)$, which is a mapping $u \to \varphi(u)$ of P into E such that

$$\varphi(R_g(u)) = \rho^{-1}(g)\varphi(u) ,$$

which we shall consider as a 0-form of type $\rho(G)$.

Given a tensorial p-form φ on P of type $\rho(G)$ a p-form on M can be defined as follows. Let $\{U_\alpha\}$ be an open covering of M by coordinate neighborhoods, and $\psi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ diffeomorphisms with corresponding transition functions $\psi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$. For each U_α, we define

$$(1) \qquad \begin{aligned} \varphi_\alpha(X_1, \cdots, X_p) &= \rho(\sigma_\alpha(u))\varphi(X_1^*, \cdots, X_p^*) , \\ \psi_\alpha(u) &= (\pi(u), \sigma_\alpha(u)) , \end{aligned}$$

where $X_j \in T_x(M)$, X_j^* is the unique horizontal lift of X_j to $u \in \pi^{-1}(U_\alpha)$, $j = 1, \cdots, p$, and $\pi(u) = x$. We see immediately that for $x \in U_\alpha \cap U_\beta$,

$$\varphi_\alpha(X_1, \cdots, X_p) = \rho(\psi_{\alpha\beta})\varphi_\beta(X_1, \cdots, X_p) .$$

Conversely, if for a given coordinate covering $\{U_\alpha\}$ of M with corresponding transition functions $\psi_{\alpha\beta}$ there exist local forms φ_α with values in E satisfying (1), then a tensorial p-form φ on P of type $\rho(G)$ is determined. For example, if for a given covering $\{U_\alpha\}$ of M a connection Γ is defined by its 1-forms $\{\omega_\alpha\}$, then the curvature forms defined by

$$\Omega_\alpha = d\omega_\alpha + \tfrac{1}{2}[\omega_\alpha, \omega_\alpha]$$

determine a tensorial 2-form on P of type ad G with values in the Lie algebra of G.

In general, the exterior differential of a p-form does not preserve its tensorial character. However, the covariant differential does and is defined as follows. Let φ be a p-form on P with values in E. The covariant differential $\nabla\varphi$, with respect to a given connection Γ on P is a $(p+1)$-form defined by

$$\nabla\varphi(X_1, \cdots, X_{p+1}) = d\varphi(hX_1, \cdots, hX_{p+1}) ,$$

where d is the exterior differential operator and hX_i, $i = 1, \cdots, p+1$, denotes the horizontal component of $X_i \in T_u(P)$ with respect to the connection Γ.

If φ is a p-form of type $\rho(G)$, then $\nabla\varphi$ is a tensorial $(p+1)$-form of the same type. For example, the connection form ω of Γ on P is a 1-form of type ad G, and

$$\Omega = \nabla\omega \tag{2}$$

is a tensorial 2-form of the same type defining the curvature form of Γ. The Bianchi identity gives

$$\nabla\Omega = 0 \ . \tag{3}$$

The local forms $\nabla\varphi_\alpha$ of $\nabla\varphi$, corresponding to a covering $\{U_\alpha\}$ of M, are given by

$$(\nabla\varphi)_\alpha = d\varphi_\alpha + \tilde{\rho}(\omega_\alpha) \wedge \varphi_\alpha \ , \tag{4}$$

where $\tilde{\rho}$ is the representation of the Lie algebra of G in E, induced by ρ, and the ω_α are the connection forms on M corresponding to the given covering.

From now on, $P(M, G)$ will be the bundle of frames with structure group $G = GL(n, R)$, the general linear group over the reals R, where $n = \dim M$, and $E = R^n$. The canonical or solder form η of P is the R^n-valued 1-form on P defined by

$$\eta(X) = u^{-1}\pi(X)$$

for $X \in T_u(P)$, where the frame $u \in P$ is considered as a linear mapping $u: R^n \to T_{\pi(u)}(M)$. The form η is a tensorial 1-form on P with values in R^n, and the torsion of the connection Γ is assumed to be zero, i.e.,

$$\nabla\eta = 0 \ . \tag{5}$$

If φ^i, $i = 1, \cdots, n$, are the components of φ_α, and (ω^i_j), (Ω^i_j) the matrices of $\omega_\alpha, \Omega_\alpha$ respectively, then formulas (2) and (4) become

$$\Omega^j_i = -d\omega^j_i + \omega^k_i \wedge \omega^j_k \ , \tag{6}$$

$$(\nabla\varphi)^i = d\varphi^i + \omega^i_j \wedge \varphi^j \ . \tag{7}$$

Moreover,

$$(\nabla^2\varphi)^i = -\Omega^i_j \wedge \varphi^j \ . \tag{8}$$

(The summation convention is employed here and in the sequel.)

If f is a scalar-valued q-form on M, then by applying (7)

$$(\nabla(\varphi \wedge f))^i = \nabla\varphi^i \wedge f + (-1)^p\varphi^i \wedge df \ . \tag{9}$$

3. Tensorial p-jet forms

In the following by a tensor p-form on M of type $\rho(G)$ we will understand the forms defined on M by a tensorial p-form on P of type $\rho(G)$, as given by

(1). It is easy to see that the tensor p-forms of type $\rho(G)$ on M define a module $\mathscr{T}^p$ over the ring $\mathfrak{F}$ of differentiable functions on M, and (8) shows that the p-forms $\{\nabla^2 T\}$ define an $\mathfrak{F}$-submodule $\mathscr{D}^p$ of $\mathscr{T}^p$.

A *tensorial p-jet form* of type $\rho(G)$ on M is a pair (T, S) of tensor forms of type $\rho(G)$ and of degrees p and $p+1$, respectively [1]. Let J^p denote the $\mathfrak{F}$-module of these forms, and let K^p be the submodule of J^p defined by the jet-forms (T, S) with $S \in \mathscr{D}^{p+1}$. If M is a Riemannian manifold of constant curvature, the modules K^p for $p = 1, \cdots, n-1$ are isomorphic with the modules L^p defined by the pairs (λ, α), where λ is an R^n-valued tensor p-form and α is a scalar p-form [2]. More generally, instead of Ω one may consider a k-form Θ on an n-dimensional manifold M which is locally expressible as $dy^1 \wedge \cdots \wedge dy^k$, and tensorial jet-forms (T, S) defined in an analogous manner. In particular, the curvature form of a manifold of constant curvature has this local representation.

Let $\tilde{L}^p$ denote the submodule of L^p defined by those elements $(\lambda, \alpha) \in L^p$ such that $\nabla^2 \lambda = 0$. Note that $\tilde{L}^p = L^p$ for $p = n-1, n$, and that $(\eta \wedge \varphi, \alpha) \in \tilde{L}^p$ for any scalar-valued $(p-1)$-form φ and p-form α on M. We define an operator D^p on $\tilde{L}^p$ as follows:

$$D^p(\lambda, \alpha) = (\nabla \lambda - \eta \wedge \alpha, d\alpha) . \tag{10}$$

Clearly, $D^p\colon \tilde{L}^p \to \tilde{L}^{p+1}$, and from (10) we have $D^{p+1} \circ D^p = 0$. In the sequel, we shall occasionally write D for D^p, $p = 0, 1, \cdots, n$.

A multiplication between the elements of $\tilde{L} = \oplus_{p=0}^n \tilde{L}^p$ is defined as follows:

$$(\lambda, \alpha) \times (\mu, \beta) = (\lambda \wedge \beta + \alpha \wedge \mu, 2\alpha \wedge \beta) , \tag{11}$$

where $(\lambda, \alpha) \in \tilde{L}^p$, $(\mu, \beta) \in \tilde{L}^q$. Clearly, $(\lambda, \alpha) \times (\mu, \beta) \in \tilde{L}^{p+q}$, and we have

$$(\lambda, \alpha) \times (\mu, \beta) = (-1)^{pq}(\mu, \beta) \times (\lambda, \alpha) .$$

A simple computation shows that

$$D[(\lambda, \alpha) \times (\mu, \beta)] = D(\lambda, \alpha) \times (\mu, \beta) + (-1)^p(\lambda, \alpha) \times D(\mu, \beta) .$$

Thus $\tilde{L}$ is a graded ring, and D is a derivation on $\tilde{L}$.

Note that (i) if one of the factors (λ, α), (μ, β) is D-closed, then the product is D-closed; (ii) if one of the factors is D-closed and the other is D-exact, then the product is D-exact.

Consider the cochain complex

$$\tilde{L} = \left(\bigoplus_{p=0}^{n} \tilde{L}^p, D^p\right) ,$$

and assume that the Poincaré lemma for D holds, viz., on an open ball in R^n

every D-closed element of $\tilde{L}^p$, $p > 0$, is D-exact. This is certainly the case if M is locally flat. On the other hand, if we consider the submodules of $\tilde{L}^p$, $p \leq n-1$, consisting of the pairs $(\eta \wedge \varphi, \alpha)$, the Poincaré lemma is again valid. The cohomology groups

$$H^p(\tilde{L}) = \operatorname{Ker} D^p / \operatorname{Im} D^{p-1}\,, \qquad p = 1, \cdots, n\,, \tag{12}$$

will be called the *curvature groups of the connection* Γ. We shall also write $H^p(M, \Gamma)$ for $H^p(\tilde{L})$, and define $H^0(M, \Gamma)$ to be Ker D^0.

4. *s*-fields

Suppose now that the manifold M is pseudo-Riemannian with metric γ. The system of first order partial differential equations

$$\nabla_j X^k = f\delta_j^k\,, \tag{13}$$

where δ_j^k is the Kronecker delta and f is a C^∞ function, defines an infinitesimal conformal transformation X of (M, γ). This system may be written in the form

$$\nabla_j \xi_i = f\gamma_{ij}\,, \tag{14}$$

where $\xi_i = \gamma_{ik}X^k$. The 1-form $\xi = \xi_i dx^i$ defined by duality in terms of the metric is therefore closed. Hence by the Poincaré lemma ξ is (locally) the gradient of a function. The (special) infinitesimal conformal transformations characterized by (13) will be called *s-fields*. The s-fields define an additive abelian group S but not an $\mathfrak{F}$-module.

The integrability conditions of (13) yield

$$X^r R^i{}_{rjk} = \nabla_k f\delta_j^i - \nabla_j f\delta_k^i\,, \tag{15}$$

where $\Omega_j^i = R^i{}_{jkl}dx^k \wedge dx^l$. Contracting (15) gives

$$X^r R_{rj} = -(n-1)\nabla_j f\,, \tag{16}$$

where $R_{jk} = R^i{}_{jki}$ is the Ricci tensor of (M, γ). Substituting (16) in (15), we get

$$X^r W^i{}_{rjk} = 0\,, \tag{17}$$

where the tensor field

$$W^i{}_{jkl} = R^i{}_{jkl} - \frac{1}{n-1}(R_{jk}\delta_l^i - R_{jl}\delta_k^i)$$

is the Weyl projective curvature tensor. Thus (17) gives a necessary condition for (13) to have a solution. In particular, this condition is satisfied if (M, γ) is projectively flat.

In the sequel, we will be particularly interested in the case where $f =$ constant in (13). In this case, from (15)

$$X^r R^i{}_{rjk} = 0 \,, \tag{18}$$

which is satisfied if γ is Ricci flat, as can be seen from (16). From (16) we see that if f is constant and the Ricci tensor is nondegenerate at each point of M, then there are no nontrivial solutions of the system (13). The vector fields satisfying (13) with $f =$ constant are infinitesimal homothetic transformations.

5. Cohomology with coefficients in the sheaf of germs of s-fields

Let $\tilde{S}$ be the subspace of s-fields characterized as solutions of (13) with $f = c$ (constant) which we shall call *homothetic s-fields*. There is a monomorphism

$$i: \tilde{S} \to \tilde{L}^0$$

given by $i(X) = (X, c)$. Let $\mathscr{S}$ be the sheaf of germs of homothetic s-fields of M and $\mathscr{L}^p, p \geq 0$, the sheaves of germs associated with the modules $\tilde{L}^p$. The mapping $D: \tilde{L}^p \to \tilde{L}^{p+1}$ induces a mapping $\mathscr{L}^p \to \mathscr{L}^{p+1}$ which we again denote by D. We then have a sequence of sheaf homomorphisms

$$0 \longrightarrow \mathscr{S} \xrightarrow{\;i\;} \mathscr{L}^0 \xrightarrow{\;D\;} \mathscr{L}^1 \xrightarrow{\;D\;} \cdots \xrightarrow{\;D\;} \mathscr{L}^n \longrightarrow 0 \,. \tag{19}$$

This sequence is exact. Exactness at $\mathscr{L}^0$ is clear. In fact, if $(X, f) \in \mathscr{L}^0$ and $D(X, f) = (\nabla X - \eta f, df) = 0$, then $df = 0$ and $\nabla X = f\eta$ which imply $f = c$ and $\nabla X = c\eta$, i.e.,

$$\nabla_j X^i = c\delta^i_j \,.$$

Hence $(X, f) = i(X)$. Exactness at $\mathscr{L}^p$, $p > 0$, is a consequence of the Poincaré lemma for D. The $\mathscr{L}^p, p = 0, 1, \cdots, n$, being fine sheaves the sequence (19) gives a fine resolution of $\mathscr{S}$. Hence we obtain

Theorem 1. *The curvature groups of a (pseudo-) Riemannian manifold are isomorphic with the cohomology groups of the space with coefficients in the sheaf of germs of homothetic s-fields.*

Corollary 1. *The curvature groups of a (pseudo-) Riemannian manifold whose Ricci tensor is nondegenerate everywhere are trivial.*

Corollary 2. *The curvature groups of an Einstein space with nonvanishing scalar curvature vanish. Hence an Einstein space with at least one nonvanishing curvature group is Ricci flat.*

The proof of Corollary 1 follows immediately from the last paragraph of § 4, and Corollary 2 is a consequence of Corollary 1.

If the scalar curvature is a nonzero constant, it is an easy consequence of (16) that the system

$$\nabla_j X^i = c\delta^i_j$$

cannot have a solution except possibly when $c = 0$. Hence

Theorem 2. *The curvature groups of a Riemannian manifold with constant nonzero scalar curvature are isomorphic with the cohomology groups of the manifold with coefficients in the sheaf of germs of its parallel vector fields.*

Remark. If the Ricci tensor is nondegenerate everywhere, then a D-closed 1-form (λ, α) can be expressed as $(-f\eta, df)$ for some C^∞ function f. For, by Corollary 1, $(\lambda, \alpha) = D(X, f) = (\nabla X - f\eta, df)$. But $\nabla^2 X = 0$ which by (8) implies $X^i R_{ijkl} = 0$, and by contraction $X^i R_{ij} = 0$, from which X is zero.

6. Relation between the curvature groups and de Rham groups

Let Σ^p be the $\mathfrak{F}$-module of vector-valued forms of the type $\eta \wedge \varphi$, where φ is a scalar-valued $(p-1)$-form. The covariant differential $\nabla\colon \Sigma^p \to \Sigma^{p+1}$ is then given by

$$\nabla(\eta \wedge \varphi) = -\eta \wedge d\varphi\,, \tag{20}$$

and it is a trivial fact that $\nabla^2(\eta \wedge \varphi) = \nabla(\nabla(\eta \wedge \varphi)) = 0$.

Consider the cochain complex $\Sigma = (\oplus_{p=1}^n \Sigma^p, \nabla^p)$, where $\nabla^p = \nabla\colon \Sigma^p \to \Sigma^{p+1}$, and let

$$H^p(\Sigma) = \operatorname{Ker} \nabla^p / \operatorname{Im} \nabla^{p-1}$$

denote its p-th cohomology group. Define $H^1(\Sigma) = \operatorname{Ker} \nabla^1$.

The correspondence $\varphi \to \eta \wedge \varphi$ establishes a $1-1$ mapping of the module of p-forms on M onto Σ^{p+1}, $p = 0, 1, \cdots, n-1$. It is easy to see from (20) that under this mapping d-closed forms are mapped into ∇-closed forms, and d-exact forms into ∇-exact forms.

A multiplication "$\cdot$" between the elements of Σ is defined by

$$(\eta \wedge \varphi)\cdot(\eta \wedge \psi) = \eta \wedge \varphi \wedge \psi \in \Sigma^{p+q-1}\,,$$

where φ and ψ are scalar-valued $(p-1)$- and $(q-1)$-forms, respectively. It is easily seen that

$$(\eta \wedge \varphi)\cdot(\eta \wedge \psi) = (-1)^{pq}(\eta \wedge \psi)\cdot(\eta \wedge \varphi)\,,$$

$$\nabla[(\eta \wedge \varphi)\cdot(\eta \wedge \psi)] = \nabla(\eta \wedge \varphi)\cdot\eta \wedge \psi + (-1)^{p-1}\eta \wedge \varphi\cdot\nabla(\eta \wedge \psi)\,.$$

Thus Σ is a graded ring, and ∇ is a derivation on Σ.

Lemma 1. *The p-dimensional de Rham cohomology groups of M are isomorphic with the groups $H^{p+1}(\Sigma)$, $p = 0, 1, \cdots, n-1$. Moreover, their cohomology rings are also isomorphic.*

The group $\tilde{S}$ of homothetic s-fields may also be characterized as solutions of

$$\nabla X = c\eta \ . \tag{21}$$

Note that if (21) has a solution for some $c \neq 0$, then it has a solution for every $c \in R$. We therefore have a sequence of homomorphisms

$$0 \xrightarrow{i} \tilde{S} \xrightarrow{\nabla} \Sigma^1 \xrightarrow{\nabla} \Sigma^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Sigma^n \longrightarrow 0 \ . \tag{22}$$

As before, let $\mathscr{S}$ be the sheaf of germs of homothetic s-fields of M, and let $\mathscr{S}^p, p = 1, \cdots, n$, denote the sheaves of germs associated with Σ^p. The sequence (22) induces the sequence of sheaf homomorphisms

$$0 \xrightarrow{i} \mathscr{S} \xrightarrow{\nabla} \mathscr{S}^1 \xrightarrow{\nabla} \mathscr{S}^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathscr{S}^n \xrightarrow{\nabla} 0 \ . \tag{23}$$

Lemma 2. *Let (M, γ) be a (pseudo-) Riemannian manifold. If* (21) *has a solution for some $c \neq 0$ but no nonzero solution for $c = 0$ (locally), then the sequence* (23) *is exact.*

Proof. Exactness at $\mathscr{S}$ follows from the assumption that there are no parallel vector fields. Now let $f\eta \in \Sigma^1$ be ∇-closed; then $\nabla(f\eta) = -\eta \wedge df$ implies $f = c \neq 0$ (for, otherwise $f\eta = 0$). By hypothesis, there exists an $X \in \mathscr{S}$ such that $\nabla X = c\eta$. Let $\eta \wedge \varphi$ be a ∇-closed form in $\Sigma^p, p \leq n-1$. Then $\nabla(\eta \wedge \varphi) = -\eta \wedge d\varphi = 0$ implies $d\varphi = 0$, so by the Poincaré lemma $\varphi = d\sigma$, locally. Hence $\eta \wedge \varphi = -\nabla(\eta \wedge \sigma)$.

Since the sheaves $\mathscr{S}^p, p = 1, \cdots, n$, are fine, the sequence (23) gives a fine resolution of $\mathscr{S}$ under the assumptions of Lemma 2.

Theorem 3. *Under the assumptions of Lemma* 2 *the groups $H^{p+1}(\Sigma)$ are isomorphic with the cohomology groups $H^p(M, \mathscr{S})$, $p = 1, \cdots, n-1$.*

Theorem 3 together with Theorem 1 yields

Corollary 3. *Under the assumptions of Lemma* 2, *the groups $H^{p+1}(\Sigma)$ are isomorphic with the curvature groups $H^p(M, \Gamma)$, $p = 1, \cdots, n-1$.*

Corollary 3 and Lemma 1 give

Corollary 4. *Under the assumptions of Lemma* 2, *the curvature groups $H^p(M, \Gamma)$ are isomorphic with the p-dimensional de Rham groups, $p = 1, \cdots, n-1$.*

Corollary 4 also follows in a strightforward manner from Theorem 1 by observing that under the assumptions of Lemma 2, the sheaf $\mathscr{S}$ is isomorphic to the sheaf of real constants. In fact, for a germ $X \in \mathscr{S}$ we get a unique constant c from $\nabla X = c\eta$. On the other hand, for any $c \in R$ the germ X such that $\nabla X = c\eta$ is unique, since $\nabla X_i = c\eta$, $i = 1, 2$, implies $\nabla(X_1 - X_2) = 0$.

7. Concluding remarks

The curvature groups of type $\rho(G)$ as defined in [2] are the cohomology groups of the sequence of p-jet forms $\{(T, S)\}$, $S = -\Omega \wedge Q = \nabla^2 Q$, where T is a tensor p-form and Q is a tensor $(p-1)$-form. Thus S belongs to the "ideal generated by curvature". There is a chain operator $D\colon (T, S) \to (\nabla T - S, \nabla^2 T - \nabla S)$.

Another definition of this cohomology may be given as follows. Consider the quotient module $\mathscr{T}/\mathscr{D}$, where $\mathscr{T}^p$ is the module of tensor p-forms of type $\rho(G)$ and $\mathscr{D}^p = \nabla^2 \mathscr{T}^{p-2} = -\Omega \wedge \mathscr{T}^{p-2}$. Since $\nabla\Omega = 0$, $\mathscr{D}$ is invariant under ∇, so $\mathscr{T}/\mathscr{D}$ is operated on by ∇ with $\nabla^2 = 0$. We claim that $(T, S) \to T + \mathscr{D}$ is a chain map which induces an isomorphism of the cohomology of $\mathscr{T} \oplus \mathscr{D}$ onto the cohomology of $\mathscr{T}/\mathscr{D}$.

In the special case where $\mathscr{T}$ is the module of tangent vector-valued forms, there are two other chain maps connecting $\mathscr{T}/\mathscr{D}$ with the de Rham complex $\bigwedge$. These are $\bigwedge^p \to \mathscr{T}^{p+1}/\mathscr{D}$ given by $\varepsilon(\eta)$, where $\varepsilon(\eta)$ denotes exterior multiplication by the solder form η, and the alternating operator $\mathscr{A}\colon \mathscr{T}^p/\mathscr{D} \to \bigwedge^{p+1}$. The curvature identity shows that $\mathscr{A}\mathscr{D} = 0$ so that the map $\mathscr{T}/\mathscr{D} \to \bigwedge$ is well-defined.

The operator $\varepsilon(\eta)$ raises the degree of the tensor and the degree of the coefficient form by 1. It is a chain map since torsion is zero, i.e., $\nabla\eta = 0$. (There are similar chain maps, for other degrees, of tensor forms other than those of degree 1.)

As for the alternating operator $\mathscr{A}$, we can skew-symmetrize with respect to it and the form indices, thereby getting a chain map which raises the form degree by 1 and lowers the tensor degree by 1.

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